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Solving Linear and Quadratic Quaternion Equations

James D. Turner*
Dynacs Military and Defense, Houston, Texas 77058

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I. Introduction

MANY problems in algebra and differential equations require solutions for polynomial equations as necessary conditions for solving scientific and engineering problems. Solutions are readily obtained for linear and quadratic equations when the problem coefficients are real or complex. Quaternion [1–9] problems are much more challenging for two reasons: 1) quaternions are noncommutative under multiplication, and 2) four nonlinear coupled algebraic equations define the necessary conditions for the solution. Only real-valued quaternion coefficients solutions are considered [3]. Scalar linear equations have previously been handled by introducing a 4×4 matrix representation [7,8]. This paper introduces scalar algorithms for handling both linear and quadratic equations, eliminating the need for a 4×4 matrix representation. Many special cases are considered [7–9]. It is somewhat surprising that matrix algorithms exist for quaternions [10], yet the most basic quaternion solutions presented here have been unavailable for engineering applications of quaternion methods for dynamics and control problems [1,2].

Hamilton [4] developed quaternions in the 1840s as hypercomplex generalizations of complex numbers. Quaternions consist of both scalar and vector parts requiring special algebraic rules for performing calculations. The noncommuntivity of quaternion multiplication complicates all solution algorithms. The main contribution of this paper is the demonstration that the scalar and vector parts of the solution are analytically uncoupled by the introduction of a exact change-of-variables substitution, resulting in a scalar polynomial equation in a single variable that must be solved. The advantage of this approach, over a classical resultant-based solution strategy, is that a 78th-order polynomial is replaced with a second- through eighth-order polynomial, depending of the symmetries of the specific application considered.

II. Quaternion Math

The quaternion coefficients considered in this paper are expressed as $\mathbf{a} = a_0 + a_1\hat{\imath} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}} = a_0 + \boldsymbol{a}$, where $\hat{\imath}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ denote hypercomplex unit vector directions. The unit vectors are subject to the quaternion operational rules $\hat{\imath}^2 = \hat{\mathbf{j}}^2 = \hat{\mathbf{k}}^2 = \hat{\imath}\,\hat{\mathbf{j}}\,\hat{\mathbf{k}} = -1$, leading to the addition and subtraction rules given by $\mathbf{a}+\mathbf{b} = (a_0+b_0, \boldsymbol{a}+\boldsymbol{b})$ and $\mathbf{a}-\mathbf{b} = (a_0-b_0, \boldsymbol{a}-\boldsymbol{b})$. Multiplication is defined by $\mathbf{ab} = (a_0b_0 - a_v\cdot b_v, a_0b_v + b_0a_v + a_v x b_v)$, where $(^*)_0$ denotes the scalar part of a quaternion, $(^*)_v$ denotes the vector part of a quaternion , "·" denotes the vector dot product, and "×" denotes the vector cross product (i.e., this is the source of quaternion noncommuntivity).

III. Linear Quaternion Equation: $\mathbf{ax}+\mathbf{xb}+\mathbf{c}=\mathbf{0}$

Scalar linear equations are considered in this section. Meister and Schaeben [7,8] have demonstrated that scalar quaternion equations can be recast in matrix–vector form as

$$\begin{gathered}\mathbf{ax}+\mathbf{xb}=\mathbf{c}, \qquad [\mathbf{A}]\mathrm{vec}(\mathbf{x}) + [\mathbf{X}]\mathrm{vec}(\mathbf{b}) = \mathrm{vec}(\mathbf{c})\\ [\mathbf{A}]\mathrm{vec}(\mathbf{x}) + [\mathbf{B}']\mathrm{vec}(\mathbf{x}) = \mathrm{vec}(\mathbf{c})\\ \underbrace{\mathrm{vec}(\mathbf{x})}_{4\times1} = \underbrace{\{[\mathbf{A}]+[\mathbf{B}']\}^{-1}}_{4x4}\underbrace{\mathrm{vec}(\mathbf{c})}_{4\times1}\end{gathered} \tag{1}$$

where the $[^*]$ denote matrix representations of the quaternions, $[^{*\prime}]$ denotes matrix representation of a quaternion that accounts for a switch in the product order, and vec$(^*)$ denotes a vector representation of the quaternion. This approach is reminiscent of numerical strategies for solving matrix Lyapunov equations using Kronecker product algorithms [11]. The solution algorithms presented in this paper replace the 4×4 matrix linear equation defined by Eq. (1) with a scalar algorithm.

A. Linear Equation Necessary Conditions

The necessary conditions for the linear equation are obtained by invoking the addition and product rules, yielding

$$\begin{pmatrix} (a_0+b_0)x_0 - (a_v+b_v)\cdot x_v + c_0, \\ (a_0+b_0)x_v - (a_v+b_v)x_0 + \widetilde{a_v - b_v}\cdot x_v + c_v \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{0}\end{pmatrix} \tag{2}$$

where "∼" denotes the matrix form for the vector cross product given by

$$\tilde{\eta} = \begin{bmatrix} 0 & -\eta_3 & \eta_2 \\ \eta_3 & 0 & -\eta_1 \\ -\eta_2 & \eta_1 & 0 \end{bmatrix} \tag{3}$$

Several special cases are considered including: 1) $\mathbf{a}=\mathbf{b}$, 2) $\mathbf{a}\neq\mathbf{b}$, and 3) $\mathbf{c}=0$

1. *Case 1:* $\mathbf{a}=\mathbf{b}$

The necessary condition of Eq. (2) reduces to

$$\begin{pmatrix} 2a_0x_0 - 2\boldsymbol{a}\cdot\boldsymbol{x} + c_0, \\ 2a_0\boldsymbol{x} + 2\boldsymbol{a}x_0 + 0\cdot\boldsymbol{x} + \boldsymbol{c} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{0}\end{pmatrix} \tag{4}$$

where the cross product term vanishes. Manipulating the scalar and vector parts of the equation, one obtains

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*Manager, Business Development, 1300 Hercules Suite 205; currently Adjunct Associate Professor, Texas A&M University, Department of Aerospace Engineering, College Station, Texas 77843. Senior Member AIAA.

$$x_0 = -\frac{c_0 a_0 + \boldsymbol{a}\cdot\boldsymbol{c}}{2(a_0^2 + \boldsymbol{a}\cdot\boldsymbol{a})}; \qquad \boldsymbol{x} = \frac{-(2\boldsymbol{a}x_0 + \boldsymbol{c})}{2a_0} \tag{5}$$

Assuming that $\mathbf{a} \neq 0$, the scalar and vector parts of the solution always exists. A special case arises when $a_0 = 0$, where no solutions exist for equations of the form $\boldsymbol{a}\mathbf{x} + \mathbf{x}\boldsymbol{a} + \mathbf{c} = 0$; unless $a_v = \pm c_v$, $x_0 = \mp 1/2$, c_0 is arbitrary, and x_v must satisfy the constraint $\pm 2c_v \cdot x_v - c_0 = 0$.

a. Case 1.a: $\mathbf{a} = \mathbf{b}$ *and* $\mathbf{c} = 0$. The necessary condition of Eq. (2) reduces to

$$\begin{pmatrix} 2a_0x_0 - 2\boldsymbol{a}\cdot\boldsymbol{x} \\ 2a_0\boldsymbol{x} + 2\boldsymbol{a}x_0 \end{pmatrix} = \begin{bmatrix} 2a_0 & -2\boldsymbol{a}^t \\ 2\boldsymbol{a} & 2a_0 I_{3\times 3} \end{bmatrix} \begin{pmatrix} x_0 \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{0} \end{pmatrix} \tag{6}$$

which represents a classical eigenvector problem for the null space of an operator. A solution for $\mathbf{x}$ exists if and only if Eq. (6) has a zero eigenvalue, where the solution for $\mathbf{x}$ is the corresponding eigenvector. The eigenvalues of Eq. (6) can be shown to be $2a_0| - 2i\sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}; 2a_0 + 2i\sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}; 2a_0; 2a_0$. The first two are complex and are discarded as nonphysical solutions. The last two represent a double root. For the case of $a_0 = 0$ the eigenvectors for the null space quaternion directions are given by $\mathbf{x}_1 = [0, \hat{\imath} - (a_1/a_3)\hat{\mathbf{k}}]$ and $\mathbf{x}_2 = [0, \hat{\mathbf{j}} - (a_2/a_3)\hat{\mathbf{k}}]$. The solution of $\mathbf{x}_1$ is new, and the solution for $\mathbf{x}_2$ had previously been found by Meister and Schaeben [7,8].

b. Case 1.b: $\mathbf{b} = -\mathbf{a}$ *and* $\mathbf{c} = 0$. The necessary conditions of Eq. (2) reduce to

$$\begin{pmatrix} 0 \\ \widetilde{2\boldsymbol{a}}\cdot\boldsymbol{x} \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{2\boldsymbol{a}} \end{bmatrix} \begin{pmatrix} x_0 \\ \boldsymbol{x} \end{pmatrix} \Rightarrow \mathbf{x} = x_0 + c\boldsymbol{a} \tag{7}$$

where x_0 and c are arbitrary constants; this solution had previously been found by Meister and Schaeben [7,8].

2. Case 2: $\mathbf{a} \neq \mathbf{b}$

All of the terms appearing in Eq. (2) remain. A two-part strategy is presented. First, the vector part of the necessary condition is manipulated to provide the following unique change of variables

$$\boldsymbol{x} = -[(a_0 + b_0)I_{3\times 3} + \widetilde{\boldsymbol{a} - \boldsymbol{b}}]^{-1}[\boldsymbol{c} + (\boldsymbol{a} + \boldsymbol{b})x_0] \tag{8}$$

where I denotes a 3×3 identity matrix and the vector cross product terms are accounted by using the notation of Eq. (3). Second, this equation is introduced into the scalar part of Eq. (2) and manipulated to provide

$$x_0 = \frac{-\{c_0 + (\boldsymbol{a} + \boldsymbol{b})\cdot[(a_0 + b_0)I_{3\times 3} + \widetilde{\boldsymbol{a} - \boldsymbol{b}}]^{-1}\cdot\boldsymbol{c}\}}{(a_0 + b_0) + (\boldsymbol{a} + \boldsymbol{b})\cdot[(a_0 + b_0)I_{3\times 3} + \widetilde{\boldsymbol{a} - \boldsymbol{b}}]^{-1}\cdot(\boldsymbol{a} + \boldsymbol{b})} \tag{9}$$

The normal solution process first solves Eq. (9) and introduces the result into (8) to complete the quaternion solution. Given the nonlinear coupling appearing in the preceding discussion, it is of interest to investigate opportunities for the algorithm to fail.

B. Numerical Stability Issues for the Linear Equation Solution

There are two cases where the solution defined by Eqs. (8) and (9) can fail to exist. First, the inverse matrix appearing in Eqs. (8) and (9) becomes singular. Second, the denominator of Eq. (9) vanishes. Both of these cases are considered in this section.

The first problem is investigated by studying the analytic expression for the inverse matrix appearing in Eq. (8), given by

$$[(a_0 + b_0)I_{3\times 3} + \widetilde{\boldsymbol{a} - \boldsymbol{b}}]^{-1} = \frac{[(\boldsymbol{a} - \boldsymbol{b})(\boldsymbol{a} - \boldsymbol{b})^t]/(a_0 + b_0) + (a_0 + b_0)I_{3\times 3} - (\widetilde{\boldsymbol{a} - \boldsymbol{b}})}{(\boldsymbol{a} - \boldsymbol{b})\cdot(\boldsymbol{a} - \boldsymbol{b}) + (\boldsymbol{a}_0 + \boldsymbol{b}_0)^2} \tag{10}$$

Because $\mathbf{a} \neq \mathbf{b}$ the vector expression $\boldsymbol{a} - \boldsymbol{b} \neq 0$; nevertheless, Eq. (10) behaves like a matrix-valued first-order pole as $a_0 + b_0 \Rightarrow 0$. This implies that the solution fails for the case that $a_0 + b_0 \Rightarrow 0$.

The second problem is investigated by evaluating the denominator in Eq. (9), given by

$$(a_0 + b_0) + (\boldsymbol{a} + \boldsymbol{b})\cdot[(a_0 + b_0)I_{3\times 3} + \widetilde{\boldsymbol{a} - \boldsymbol{b}}]^{-1}\cdot(\boldsymbol{a} + \boldsymbol{b}) = 0 \tag{11}$$

when $a_0 + b_0 \neq 0$. Expanding Eq. (11) establishes that the only solutions for $a_0 + b_0$ satisfying this condition are $a_0 + b_0 = \pm i\sqrt{(\boldsymbol{a} - \boldsymbol{b})\cdot(\boldsymbol{a} - \boldsymbol{b})}$, which do not represent physically meaningful quaternion solutions. As a result only the $a_0 + b_0 \Rightarrow 0$ case exists as a potential problem solution. All other meaningful solutions are obtained by first solving Eq. (9) and second introducing the result for x_0 into Eq. (8), thereby completing the solution process.

IV. Quadratic Polynomial Equation: $\mathbf{p}^2 + \mathbf{ap} + \mathbf{pb} + \mathbf{c} = 0$

Quadratic equations are more complicated than the previously considered linear equations. The solutions obtained for $\mathbf{p}$ are limited to real values: no biquaternion solutions are allowed. As in the linear equation cases in the preceding discussion, all of the polynomial cases are solved by introducing a two-step process. First, the four nonlinear equations defining the necessary conditions for the solution are reduced to a single scalar equation by analytically eliminating the vector part of the solution from the scalar equation. Second, after obtaining the scalar solutions, the vector part of the solution is obtained by introducing the scalar solution into the transformation equation for the vector part of the solution.

Two cases are of interest: 1) $\mathbf{a} = \mathbf{b}$ and 2) $\mathbf{a} \neq \mathbf{b}$. The symmetric first case is shown to have a closed-form solution that is obtained by completing the square for the polynomial. The general second case is more challenging. Experience gained from performing numerical experiments has identified the starting locations for the root searching algorithm. The roots are recovered using a combination of linear searching for sign changes in the polynomial followed by Newton's method for polishing up the initial root estimates.

A. Case 1: $\mathbf{b} = \mathbf{a}$

The quadratic polynomial is expressed as: $\mathbf{p}^2 + \mathbf{ap} + \mathbf{pa} + \mathbf{c} = 0$, which is solved by *completing the square*, leading to

Table 1 Numerical test cases

Equation	Coefficients	Quaternion solution	Solution accuracy
Linear	$\mathbf{a} = \mathbf{b}, \mathbf{c} \neq \mathbf{0}$	$\mathbf{x} = -0.6928 - 1.7320\hat{\imath} - 0.8660\hat{\mathbf{j}} - 2.0784\hat{\mathbf{k}}$	$\sim 1.0 \times 10^{-15}$
Linear	$\mathbf{a} \neq \mathbf{b}, \mathbf{c} \neq \mathbf{0}$	$\mathbf{x} = \frac{3}{2} + \frac{1}{2}\hat{\imath} + \frac{1}{2}\hat{\mathbf{j}} - \frac{3}{2}\hat{\mathbf{k}}$	Exact solution
Quadratic	$\mathbf{a} = \mathbf{b}, \mathbf{c} \neq \mathbf{0}$	$p^+ = 1.9158 + 0.7471\hat{\imath} - 5.2526\hat{\mathbf{j}} + 3.1470\hat{\mathbf{k}}$	$\sim 1.0 \times 10^{-16}$
		$p^- = -0.7611 - 0.7471\hat{\imath} - 0.5208\hat{\mathbf{j}} - 0.8376\hat{\mathbf{k}}$	
Quadratic	$\mathbf{a} \neq \mathbf{b}, \mathbf{c} \neq \mathbf{0}$	$\mathbf{p}_1 = 0.4676 - 1.9651\hat{\imath} - 0.9059\hat{\mathbf{j}} - 0.4992\hat{\mathbf{k}}$	$\sim 1.0 \times 10^{-12}$
		$\mathbf{p}_2 = 0.5323 + 0.8911\hat{\imath} - 0.3165\hat{\mathbf{j}} - 1.0935\hat{\mathbf{k}}$	

$$\mathbf{p}^2 + \mathbf{ap} + \mathbf{pa} + \mathbf{c} = 0, \qquad (\mathbf{p} + \mathbf{a})(\mathbf{p} + \mathbf{a}) + \mathbf{c} - \mathbf{a}^2 = 0 \tag{12}$$

where the solution for **p** follows as

$$\mathbf{p}^{\pm} = -\mathbf{a} \pm \sqrt{\mathbf{a}^2 - \mathbf{c}} \tag{13}$$

and the quaternion *square root* is computed as

$$\mathbf{q} = s + \boldsymbol{v}, \qquad |\mathbf{q}| = \sqrt{s^2 + \boldsymbol{v}\cdot\boldsymbol{v}}, \qquad |\boldsymbol{v}| = \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$$
$$\theta = \tan^{-1}\left(\frac{|\boldsymbol{v}|}{s}\right), \qquad \sqrt{\mathbf{q}} = |\mathbf{q}|\left\{\cos\left(\frac{\theta}{2}\right) + \frac{\boldsymbol{v}}{|\boldsymbol{v}|}\sin\left(\frac{\theta}{2}\right)\right\}$$

The solution for Eq. (13) is checked by introducing **p** into Eq. (12) and verifying that the symmetric quaternion quadratic equation is satisfied.

B. Case 2: a ≠ b

The algebraic necessary conditions for $\mathbf{p}^2 + \mathbf{ap} + \mathbf{pb} + \mathbf{c} = 0$ are obtained by applying addition and product rules and collecting terms, yielding

$$\begin{pmatrix} p_0^2 - \boldsymbol{p}\cdot\boldsymbol{p} + (a_0 + b_0)p_0 - (\boldsymbol{a}+\boldsymbol{b})\cdot\boldsymbol{p} + c_0, \\ 2p_0\boldsymbol{p} + (a_0 + b_0)\boldsymbol{p} + (\boldsymbol{a}+\boldsymbol{b})p_0 + \widetilde{\boldsymbol{a}-\boldsymbol{b}}\cdot\boldsymbol{p} + \boldsymbol{c} \end{pmatrix} = 0 \tag{14}$$

The last equation is easily manipulated to provide the *change-of-variables transformation* given by

$$\boldsymbol{p} = -[\{2p_0 + (a_0 + b_0)\}I_{3\times3} + \widetilde{\boldsymbol{a}-\boldsymbol{b}}]^{-1}[\boldsymbol{c} + (\boldsymbol{a}+\boldsymbol{b})p_0] \tag{15}$$

which is a slightly modified version of Eq. (8). Equation (15) uncouples the scalar and vector parts of the necessary conditions for the algebraic solution. A single nonlinear polynomial equation is obtained for the scalar part of the necessary condition for Eq. (14) by introducing Eq. (15) into

$$p_0^2 + (a_0 + b_0)p_0 - (a_v + b_v)\cdot\boldsymbol{p} - \boldsymbol{p}\cdot\boldsymbol{p} + c_0 = 0 \tag{16}$$

C. Numerical Solution Strategy for Polynomial Roots

Experiments with Eq. (16) indicate that the roots of the equation are located in the vicinity of $p_0 = -(a_0 + b_0)/2$, which corresponds to the singularity of the determinant of the matrix inverse appearing in Eq. (15). The singular behavior of Eq. (16) is eliminated by factoring the matrix determinant out of the inverse solution for Eq. (15). Analytically this is accomplished by rewriting the matrix inverse of Eq. (15) as

$$M = \{2p_0 + (a_0 + b_0)\}I_{3\times3} + \widetilde{\boldsymbol{a}-\boldsymbol{b}}$$
$$\boldsymbol{p} = -[M]^{-1}[\boldsymbol{c} + (\boldsymbol{a}+\boldsymbol{b})p_0] = -\frac{\text{adjoint}(M)}{\det(M)}[\boldsymbol{c} + (\boldsymbol{a}+\boldsymbol{b})p_0] \tag{17}$$

where the determinant and adjoint follow as

$$\det(M) = (2p_0 + a_0 + b_0)\{(\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}) + (2p_0 + a_0 + b_0)^2\}$$
$$\text{adjoint}(M) = (\boldsymbol{a}-\boldsymbol{b})(\boldsymbol{a}-\boldsymbol{b})^t + (2p_0 + a_0 + b_0)^2 I_{3\times3} - (2p_0 + a_0 + b_0)(\widetilde{\boldsymbol{a}-\boldsymbol{b}})$$

The determinant is only singular when $p_0 = -(a_0 + b_0)/2$. Near this point the determinant of Eq. (15) behaves as $\det(M)|_{p_0\approx-(a_0+b_0)/2} \approx 2p_0 + a_0 + b_0 \approx 0$. Introducing this result into Eq. (16) produces a quadratic singularity. The singularity is analytically eliminated from the problem by introducing Eq. (15) into Eq. (16) and multiplying the resulting equation by $\det(M)^2$, yielding the modified polynomial equation

$$\det(M)^2[p_0^2 + (a_0 + b_0)p_0 + c_0] - \det(M)(\boldsymbol{a}+\boldsymbol{b})\cdot\boldsymbol{y} - \boldsymbol{y}\cdot\boldsymbol{y} = 0$$
$$\boldsymbol{y} = -\text{adjoint}(M)[\boldsymbol{c} + (\boldsymbol{a}+\boldsymbol{b})p_0] \tag{18}$$

Equation (18) is easily handled by numerical root finding algorithms. The first root is obtained by conducting a simple line search for a sign change in the polynomial, which is assumed to occur on the kth step of the process. A linear line model is fit to the last two data points to obtain a refined estimate for the root location as,

$$p_{0,k} \approx -\frac{y_{k-1} - mp_{0,k-1}}{m}; \qquad m = \frac{y_k - y_{k-1}}{p_{0,k} - p_{0,k-1}}$$

This value is used as a starting guess for a finite difference based Newton iteration for Eq. (18), which requires 5–6 iterations for *10-place* accuracy. The roots have been found to be nearly symmetrically located about the starting guess provided by $p_0 = -(a_0 + b_0)/2$. Given a converged solution for the first root, the second root is estimated to be $p_{0,k_2} \approx 2x_0 - p_{0,k_1}$ and is refined with Newton's method. The polished root values are introduced into Eq. (15) to complete the vector part of the solution for the quaternion roots of $\mathbf{p}^2 + \mathbf{ap} + \mathbf{pa} + \mathbf{c} = 0$.

V. Numerical Results

Example applications are presented for the linear and quadratic equations presented. A simple extension is presented for developing closed-form solutions for second-order quaternion linear differential equations. The quadratic solution is used for developing a closed-form solution for a quaternion linear differential equation. The following quaternion parameters are assumed for all the applications:

$$\mathbf{a} = 1 - 2\hat{\imath} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}; \qquad \mathbf{b} = -2 + 3\hat{\imath} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$$
$$\mathbf{c} = -1 - 2\hat{\imath} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}} \tag{19}$$

The results presented in Table 1 display four digits of numerical precision; however, all calculations have been preformed to double precision accuracy to assess the accuracy of the solution methods. Table 1 presents the equation type, the assumed coefficient properties, the resulting solution, and the accuracy achieved for each component of the quaternion. The first linear case makes use of Eq. (5). The second linear case makes use of Eqs. (8) and (9). The first quadratic case makes use of Eq. (13), where the quaternion square roots algorithm following Eq. (13) has been used. The last quadratic case makes use of (18) for conducting the linear search algorithm for detecting a sign change in the polynomial. In all cases the solution accuracy achieved is very high.

VI. Conclusion

Mathematical algorithms have been presented for solving linear and quadratic quaternion equations. Simplified solution strategies have been developed by recognizing that the scalar and vector parts of the necessary conditions can be rigorously uncoupled, yielding a single scalar equation that must be solved. Both algebraic and linear differential quaternonic applications have been presented. High accuracy is achieved for all methods.

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